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Algebraic Approximation for the $E(m)$ Function: Application to Complex Nonlinear Oscillators and Geometric Problems

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Abstract
An approximate algebraic expression composed of elementary functions was derived for the computation of the complete elliptic integral of the second kind (i.e. $E(m)$ function). The algebraic expression for $E(m)$ was formulated based on the well-known arithmetic-geometric mean (AGM) and a modified arithmetic-geometric mean (MAGM). The accuracy of the algebraic expression was tested for a wide range of positive and negative m-values and the results showed that the percentage relative error in computing $E(m)$ was less than $5.1 \times 10^{-4}\%$ for $-1000 \leq m \leq 0.999$ and less than $9.3 \times 10^{-3}\%$ for $-10000 \leq m \leq 0.9999$. The algebraic expression for $E(m)$ was shown to be applicable in finding the periodic solution of some complex nonlinear oscillators and solving some geometry-related problems. The results of the cases considered proved that present algebraic expression is capable of very accurate solutions for physical problems in science and engineering that are exactly solvable in terms of the $E(m)$ function.

Keywords: *Elliptic integral, Arithmetic-geometric mean, Circumference of ellipse, Nonlinear oscillator, Periodic solution*

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1. Introduction

The periodic solution of some oscillators can be derived exactly in terms of complete elliptic integrals. For instance, the exact periods for the nonlinear pendulum and some Duffing-type oscillators are expressible in terms of the $K(m)$ function, while the exact periods for the oscillations of a particle on a rotating parabola (Nayfeh and Mook, 1995) and the relativistic oscillator (Big-Alabo et al, 2021) can be derived in terms of the $E(m)$ function. Also, the approximate periodic solution for nonlinear Hamiltonian oscillators has been derived in terms of the $K(m)$ function (Belendez et al, 2009; Big-Alabo, 2020a). Furthermore, the geometric problems of estimating the perimeters of the lemniscate and ellipse are exactly solvable in terms of the $K(m)$ and $E(m)$ functions respectively (Adlaj, 2012).

It is well-known that complete elliptic integrals are not exactly solvable in terms of elementary functions and their equivalent infinite series converge very slowly, thus requiring very many terms to compute the required accuracy. One way to evaluate the complete elliptic integrals is to use numerical algorithms. Traditional numerical integration schemes can be applied but their convergence is slow and would require many iteration steps to achieve the required accuracy for eccentricities close to the limit of one. Another approach to evaluate the complete elliptic integrals is to apply recursive algorithms such as the AGM algorithm. The connection between the $K(m)$ function and the AGM algorithm was first discovered by Carl Friedrich Gauss in 1799. This connection has proven to be a monumental and an unsurpassed innovation in the evaluation of the $K(m)$ function because the AGM algorithm only requires a few iterations to achieve the required accuracy due to its quadratic convergence. For instance, four iterations of the AGM are sufficient to get results that are accurate to more than ten decimal places in many cases (Adlaj, 2012). Hence, the connection between the AGM and the $E(m)$ function was quickly established (Abramowitz and Stegun, 1972). However, the AGM formula for the $E(m)$ function requires a third recurrence relationship which appears as a series in the final expression. To eliminate this series, Adlaj (2012) formulated a modified-AGM (MAGM) algorithm to derive an exact MAGM-AGM formula for the $E(m)$ function. The MAGM-AGM formula for the $E(m)$ function is analogous to the AGM formula for the $K(m)$ function.

In spite of the success recorded in the numerical estimation of complete elliptic integrals, the development of approximate algebraic formulae to compute these integrals has been relatively slow and not much progress has been made in this regard. Algebraic formulae are desirable and more attractive because they are faster to compute, use less computer memory and suitable for pedagogical uses. Abramowitz and Stegun (1972) presented polynomial approximations for the $K(m)$ and $E(m)$ functions when $0 \leq m < 1$ but none was presented for $m < 0$. In this paper, a new algebraic formula for the $E(m)$ function was formulated based on the MAGM-AGM formula for the $E(m)$ function and four iterations of the AGM and MAGM algorithms. The accuracy of the formula for $0 \leq m < 1$ and for $m < 0$ was evaluated, and its application to find the periodic solution of some complex oscillators and to solve some geometric problems was investigated.

2. Formulation of approximate algebraic solution for $E(m)$

The complete elliptic integral of the second kind is given by the following integral function (Abramowitz and Stegun, 1972):

$$E(m) = \int_0^1 \frac{\sqrt{1 - mt^2}}{\sqrt{1 - t^2}}\, dt \quad (1)$$

where $m = k^2$ is the elliptic parameter and k is the elliptic modulus or eccentricity. Substituting $t = \sin\theta$ in Equation (1) gives the angular form of the integral as:

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\, d\theta \quad (2)$$

It is clear from Equation (1) or (2) that $E(0) = \pi/2$ and $E(1) = 1$. Therefore, the challenge lies in estimating $E(m)$ for $0 < m < 1$ and for $m < 0$. Since the $E(m)$ function has no exact solution based on a combination of elementary functions, the aim of this article is to provide an approximate solution that is composed of elementary functions. To achieve this aim, we apply the exact expression for $E(m)$ in terms of the AGM and MAGM sequences, which was first derived by Adlaj (2012) as:

$$E(m) = \frac{\pi}{2}\left(\frac{N(1, \beta^2)}{M(1, \beta)}\right) \quad (3)$$

where $M(x, y)$ is the AGM of x and y, $N(x, y)$ is the MAGM of x and y, $\beta = \sqrt{1 - m}$ is the complementary elliptic modulus, and $x > y > 0$.

The AGM of two arbitrary numbers x and y is the common point of convergence of two recurrence sequences: one is an arithmetic sequence and the other is a geometric sequence. The value of the arithmetic mean decreases with each iteration while the value of the geometric mean increases so that both sequences converge to a common limit. Therefore, $M(x, y) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ where a_n and b_n are the arithmetic and geometric means given as:

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) \quad (4a)$$

$$b_n = \sqrt{a_{n-1} b_{n-1}} \quad (4b)$$

In evaluating Equations (3), the initial values are given as $a_0 = x$ and $b_0 = y$, whereas $n \geq 1$. Adlaj (2012) showed that four iterations of the AGM and MAGM sequences are sufficient to produce the required accuracy for most mathematical or scientific applications. Hence, the present derivation of the approximate algebraic solution is based on a fourth-term approximation of the AGM and MAGM sequences.

In Equations (4), the current approximations are determined from the immediate preceding terms of the sequences. Applying the AGM in this form results to a very lengthy expression for the fourth-term approximation that can only be comfortably implemented using spreadsheet applications or computer program (Carvalhaes and Suppes, 2008). Here, simple algebraic simplifications are applied to derive a recurrence relation that expresses the current approximation in terms of the initial values so that the fourth-term approximation can be evaluated with a pocket calculator. The procedure is as follows.

From equations (4), we can write that:

$$a_{n-1} = \frac{1}{2}(a_{n-2} + b_{n-2}) \quad (5a)$$

$$b_{n-1} = \sqrt{a_{n-2} b_{n-2}} \quad (5b)$$

Substituting equations (5) in (4) and simplifying gives:

$$a_n = \left(\frac{\sqrt{a_{n-2}} + \sqrt{b_{n-2}}}{2}\right)^2 \quad (6a)$$

and

$$b_n = \left(\frac{a_{n-2} + b_{n-2}}{2}\right)^{1/2} (a_{n-2} b_{n-2})^{1/4} \quad (6b)$$

Equations (6) express the current approximation in terms of the values of two iterations before. This implies that the second-term approximation can be obtained directly from the initial values. Similarly,

$$a_{n-2}=\left(\frac{\sqrt{a_{n-4}}+\sqrt{b_{n-4}}}{2}\right)^2 \quad (7a)$$

and

$$b_{n-2}=\left(\frac{a_{n-4}+b_{n-4}}{2}\right)^{1/2}(a_{n-4}b_{n-4})^{1/4} \quad (7b)$$

Then, substituting Equations (7) into (6a) and simplifying gives:

$$a_n=\frac{1}{16}\left[\sqrt{a_{n-4}}+\sqrt{b_{n-4}}+\left(8(a_{n-4}+b_{n-4})\sqrt{a_{n-4}b_{n-4}}\right)^{1/4}\right]^2 \quad (8)$$

From Equation (8), we can see that the fourth-term approximation can be obtained directly from the starting values by substituting $n=4$. Hence,

$$M(a_0,b_0)\cong a_4=\frac{1}{16}\left[\sqrt{a_0}+\sqrt{b_0}+\left(8(a_0+b_0)\sqrt{a_0b_0}\right)^{\frac{1}{4}}\right]^2 \quad (9)$$

Assuming $a_0=1$ and $b_0=\beta$, then the expression for the AGM function in the denominator of Equation (3) is:

$$M(1,\beta)\cong\frac{1}{16}\left[1+\sqrt{\beta}+\left(8(1+\beta)\sqrt{\beta}\right)^{1/4}\right]^2 \quad (10)$$

On the other hand, the MAGM, $N(x,y)$, of two arbitrary numbers x and y is defined as the common limit of the decreasing a_n sequence and the increasing b_n sequence such that a_n and b_n are defined by a triple recurrence relationship as follows (Adlaj, 2012):

$$a_n=\frac{1}{2}(a_{n-1}+b_{n-1}) \quad (11a)$$

$$b_n=c_{n-1}+\sqrt{(a_{n-1}-c_{n-1})(b_{n-1}-c_{n-1})} \quad (11b)$$

$$c_n=c_{n-1}-\sqrt{(a_{n-1}-c_{n-1})(b_{n-1}-c_{n-1})} \quad (11c)$$

where the starting values are $a_0=x$, $b_0=y$, $c_0=0$, and $n\geq 1$. Therefore, $N(x,y)=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$. It can be easily verified that the first and second iterations of the a_n sequence gives the same result as the AGM while the first iteration only of the b_n sequence gives the same result as the AGM. Following a similar concept of algebraic simplification, the fourth-term approximation of the MAGM was derived in terms of the starting values as:

$$a_4=\frac{1}{16}\left(a_0^{1/4}+b_0^{1/4}\right)^2\left[\left(\sqrt{a_0^{1/2}+b_0^{1/2}}+\sqrt{8(a_0b_0)^{1/4}}\right)^2-14(a_0b_0)^{1/4}\right] \quad (12)$$

The detail of the derivation of Equation (12) is shown in the appendix. Now, assuming that $a_0=1$ and $b_0=\beta^2$, then the expression for the MAGM function in the numerator of Equation (3) can be approximated as:

$$N(1,\beta^2)\cong\tfrac{1}{16}\left(1+\sqrt{\beta}\right)^2\left[\left(\sqrt{1+\beta}+\sqrt{8\beta^{1/2}}\right)^2-14\sqrt{\beta}\right] \quad (13)$$

Applying Equations (10) and (13) in Equation (3), the approximate expression for $E(m)$ in terms of elementary functions is:

$$E_{app}(m)=\frac{\pi\left(1+\sqrt{\beta}\right)^2\left[\left(\sqrt{1+\beta}+\sqrt{8\beta^{1/2}}\right)^2-14\sqrt{\beta}\right]}{2\left[1+\sqrt{\beta}+\left(8(1+\beta)\sqrt{\beta}\right)^{1/4}\right]^2} \quad (14)$$

Note that the AGM approximation derived in Equation (10) leads to an approximate algebraic formula for the $K(m)$ function as shown later in Equation (25), but the present study is focused on the $E(m)$ function.

3. Results and discussion

3.1 Accuracy of the $E(m)$ approximation

The accuracy of Equation (14) was tested for a wide range of m-values as shown in Fig. 1 and 2. All exact solutions were obtained using the EllipticE [·] function in Mathematica™. A perfect match can be seen between the exact and approximate solutions. Numerical computations revealed that the percentage relative error, i.e. $RE=100\%\left|E(m)-E_{app}(m)\right|/E(m)$, increased with the magnitude of m. For positive m-values, the RE in using Equation (14) was found to be $1.64399\times10^{-14}\%$, $2.37526\times10^{-10}\%$, $5.02444\times10^{-4}\%$ and $9.20983\times10^{-3}\%$ for $m=0.5$, 0.9, 0.999 and 0.9999 respectively. For negative m-values, the RE was found to be $4.34578\times10^{-10}\%$, $4.10948\times10^{-6}\%$, $5.03237\times10^{-4}\%$ and $9.21076\times10^{-3}\%$ for $m=-10$, -100, -1000 and -10000 respectively. These results confirm the accuracy of Equation (14) for a wide range of m-values.

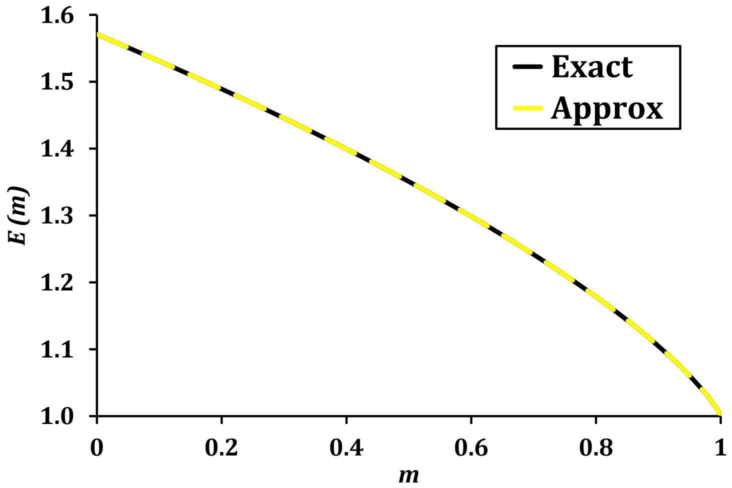

Fig. 1: Plot of the $E(m)$ function for $0 \leq m \leq 0.999$. For coloured plot see online version.

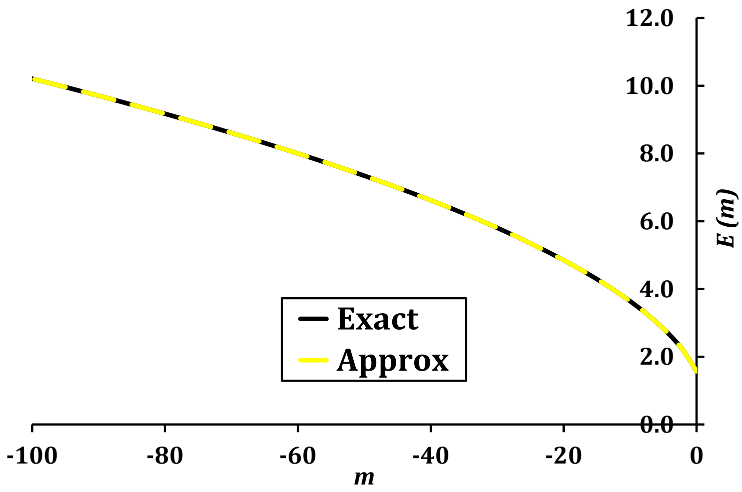

Fig. 2: Plot of the $E(m)$ function for $-100 \leq m \leq 0$. For coloured plot see online version.

Abramowitz and Stegun (1972) gave the following approximation for $0 \leq m < 1$:

$$E_{app}(m) = (1 + a_1\beta^2 + a_2\beta^4 + a_3\beta^6 + a_4\beta^8) - (b_1\beta^2 + b_2\beta^4 + b_3\beta^6 + b_4\beta^8)\ln(\beta^2) \quad (15)$$

where

$a_1 = 0.44325141463$; $a_2 = 0.06260601220$; $a_3 = 0.04757383546$; $a_4 = 0.01736506451$; $b_1 = 0.24998368310$; $b_2 = 0.09220180037$; $b_3 = 0.04069697526$ and $b_4 = 0.00526449639$. The absolute error for equation (15) was given by Abramowitz and Stegun (1972) as $\left|E(m) - E_{app}(m)\right| < 2.0 \times 10^{-8}$ but numerical checks reveal otherwise. As shown in Fig. 3, the maximum absolute error was found to be 3.67947×10^{-5} and occurred at $m = 0.40$. This implies that $\left|E(m) - E_{app}(m)\right| < 4.0 \times 10^{-5}$ for Equation (15). The absolute error only exceeded 2×10^{-8} when $m \geq 0.995$ and the accuracy improves as m gets closer to 1.0. From Fig. 3, we see that the present $E(m)$ approximation in equation (14) gives a better accuracy than Equation (15) for $0 \leq m \leq 0.99$. Moreover, Equation (15) works only for a small range of negative m-values ($RE < 1.0\%$ for $-2.4 < m \leq 0$) while Equation (14) works for a much wider range (see Fig. 2 and related discussions).

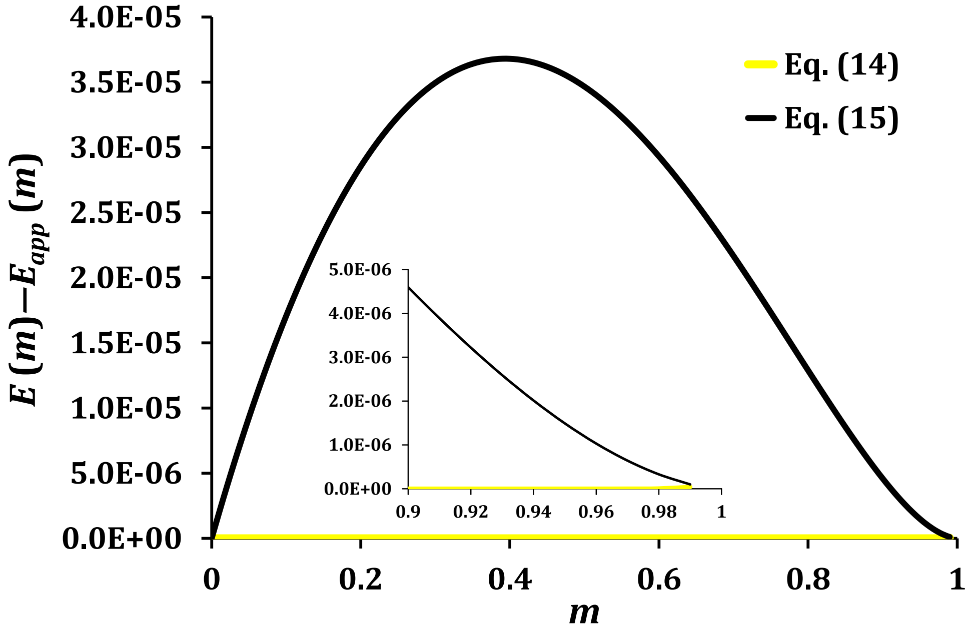

Fig. 3: Absolute error of present $E(m)$ formula and the $E(m)$ formula of Abramowitz and Stegun (1972) for $0 \leq m \leq 0.99$. For coloured plot see online version.

A simpler but less accurate algebraic expression for the $E(m)$ function can be derived based on the third-term approximations of the AGM and MAGM sequences. The resulting expression is:

$$E_{app}(m) = \frac{\pi}{2}\left[\frac{\left(1+\sqrt{\beta}\right)^2 + \sqrt{8}\left(\sqrt{\beta}+\sqrt{\beta^3}\right)^{1/2}}{(1-\beta)^2 + 4\left(\sqrt{\beta}+\sqrt{\beta^3}\right)}\right] \quad (16)$$

Equation (16) gives a maximum RE that is less than $8.0 \times 10^{-4}\%$ for $-9.0 \leq m \leq 0.9$, less than 0.10% for $-100 \leq m \leq 0.99$ and less than 1.0% for $-1000 \leq m \leq 0.999$. Hence, Equation (16) is applicable for a wide range of m-values and produces a reasonable accuracy even in extreme cases.

3.2 Periodic solutions of complex nonlinear oscillators

3.2.1 Simple harmonic relativistic oscillator

The equation governing the periodic displacement of the simple harmonic relativistic oscillator can be written in non-dimensional form as (Big-Alabo, 2020b):

$$X'' + [1 - (X')^2]^{3/2} X = 0 \quad (17)$$

with initial conditions $X(0) = 0$ and $X'(0) = V_0$; where V_0 is the maximum non-dimensional velocity which is related to the amplitude (A) of the non-dimensional displacement as follows: $V_0 = A\sqrt{4 + A^2}/(2 + A^2)$. By integrating Equation (17), the exact period was derived as:

$$T_{ex} = 4\int_0^A \frac{1 + \frac{1}{2}(A^2 - X^2)}{\sqrt{(A^2 - X^2) + \frac{1}{4}(A^2 - X^2)^2}} dX \quad (18)$$

which can be rewritten as:

$$T_{ex} = 8\int_0^A \frac{1}{(A^2 - X^2)^{1/2}[4 + (A^2 - X^2)]^{1/2}} dX + 4\int_0^A \frac{(A^2 - X^2)}{(A^2 - X^2)^{1/2}[4 + (A^2 - X^2)]^{1/2}} dX \quad (19)$$

Using the transformation $X = A\sin\theta$, Equation (19) becomes

$$T_{ex} = 8\int_0^{\pi/2} \frac{1}{(4 + A^2\cos^2\theta)^{1/2}} d\theta + 4\int_0^{\pi/2} \frac{A^2\cos^2\theta}{(4 + A^2\cos^2\theta)^{1/2}} d\theta \quad (20)$$

By simple algebraic manipulation of the second integral, we get:

$$T_{ex} = 4\int_0^{\pi/2} (4 + A^2\cos^2\theta)^{1/2} d\theta - 8\int_0^{\pi/2} \frac{1}{(4 + A^2\cos^2\theta)^{1/2}} d\theta \quad (21)$$

Then, applying the trigonometric identity $\cos^2\theta = 1 - \sin^2\theta$ gives,

$$T_{ex} = 4\sqrt{4+A^2}\int_0^{\pi/2}(1-m\sin^2\theta)^{1/2}d\theta - \frac{8}{\sqrt{4+A^2}}\int_0^{\pi/2}\frac{1}{(1-m\sin^2\theta)^{1/2}}d\theta \quad (22)$$

where $m = A^2/(4+A^2)$. The first integral of Equation (22) is the $E(m)$ function while the second integral is the $K(m)$ function. Therefore,

$$T_{ex} = 4\sqrt{4+A^2}E(m) - \frac{8}{\sqrt{4+A^2}}K(m) \quad (23)$$

The exact frequency solution is then given as:

$$\omega_{ex} = \frac{\pi\sqrt{4+A^2}}{2(4+A^2)E(m)-4K(m)} \quad (24)$$

Although the present study is on the $E(m)$ function, evaluation of the $K(m)$ function is required for equation (24). Thus, using Equation (10), an approximate formula for $K(m)$ can be derived as:

$$K_{app}(m) = \frac{\pi}{2M(1,\beta)} = \frac{8\pi}{\left[1+\sqrt{\beta}+\left(8(1+\beta)\sqrt{\beta}\right)^{1/4}\right]^2} \quad (25)$$

The maximum RE produced by Equation (25) is less than $4.0\times10^{-5}\%$ for $-1000.0 \le m \le 0.999$. Now, substituting Equations (14) and (25) in Equation (24) and simplifying, the approximate frequency (ω_{app}) solution was derived as:

$$\omega_{app} = \frac{\beta\left[1+\sqrt{\beta}+\left(8(1+\beta)\sqrt{\beta}\right)^{1/4}\right]^2}{2\left(1+\sqrt{\beta}\right)^2\left[\left(\sqrt{1+\beta}+\sqrt{8\beta^{1/2}}\right)^2-14\sqrt{\beta}\right]-16\beta^2} \quad (26)$$

Table 1: Exact and approximate frequency for the relativistic oscillator

A	V_0	ω_{ex}	ω_{app}	% RE
0.01	0.009999625	0.999981	0.999981	0.00000000
0.1	0.099626788	0.998130	0.998130	0.00000000
1	0.745355993	0.851301	0.851301	0.00000000
2	0.942809042	0.626023	0.626023	0.00000000
5	0.997252742	0.301640	0.301640	0.00000000
10	0.999807748	0.155498	0.155498	0.00000000
20	0.999987624	0.0783428	0.0783428	0.00000000
40	0.999999221	0.0392453	0.0392453	0.00000000
60	0.999999846	0.0261727	0.0261725	0.00076416
80	0.999999951	0.0196319	0.0196317	0.00101875
100	0.999999980	0.0157064	0.0157061	0.00191005

where $\beta = 2/\sqrt{4+A^2}$.

Equation (26) was used to compute the frequency of the relativistic oscillator and the results were compared with the exact solution as shown in Table 1. The maximum RE of the approximate frequency for $0 < A \le 100$ was calculated to be $1.91\times10^{-3}\%$ at $A = 100$. This error is several orders less than other published approximate solutions (Big-Alabo et al, 2021; Big-Alabo, 2020b). The A-values considered in Table 1 ranged from the non-relativistic regime (where the oscillator is essentially behaving as a simple harmonic oscillator) to the ultra-relativistic regime (where the oscillator behaves like a light photon).

3.2.2. Particle on a rotating parabola

Let us consider the free oscillations of a particle on a rotating parabola as shown in Figure 4. The system consists of a frictionless mass, m, sliding along a vertical parabolic wire whose curvature can be described as $y = qu^2$ where $q > 0$ is a constant that determines the curvature of the wire. The wire rotates at a constant speed, Ω, about the y-axis. The u-axis, which is the system's degree of freedom, is considered to be perpendicular to the vertical axis.

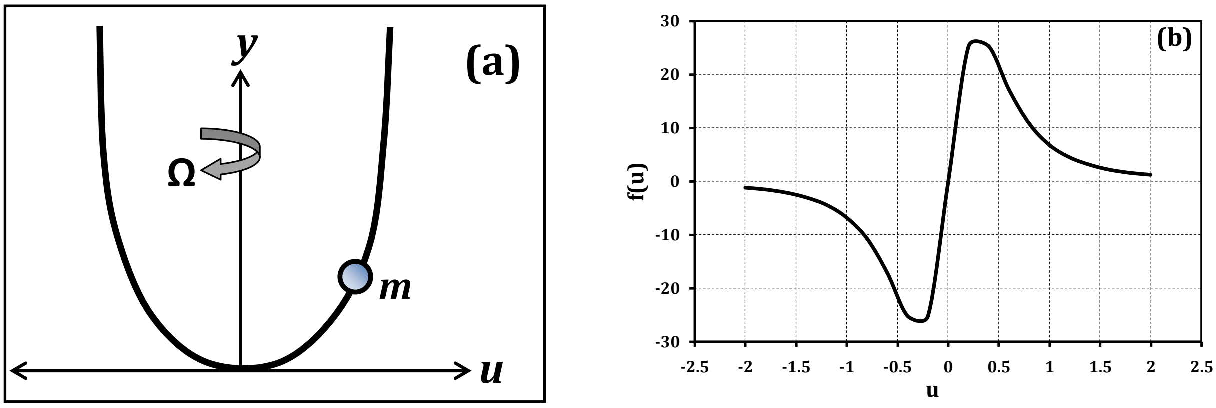

Fig. 4: (a) Schematic of particle on a rotating parabola (b) Restoring force when $q = 1.0$; $\Lambda = 10.0$ and $A = 2.0$ (Source: Big-Alabo and Ossia, 2020).

Table 2: Exact and approximate frequency for particle on a rotating parabola ($\Lambda = 10.0; q = 1.0$)

		ω_{app}			
A	ω_{ex}	CPLM	% RE	Eq. (32)	% RE
0.01	3.16196	3.16228	0.01012031	3.16196	4.21342×10^{-14}
0.1	3.13120	3.13415	0.09421308	3.13120	2.83665×10^{-14}
0.5	2.60054	2.6023	0.06767825	2.60054	3.41536×10^{-14}
1.0	1.88499	1.88541	0.02228129	1.88499	1.03661×10^{-12}
2.0	1.12710	1.12761	0.04524887	1.12710	5.03113×10^{-9}
5.0	0.486548	0.486815	0.05487639	0.486548	4.10948×10^{-6}
10	0.246859	0.246978	0.04820566	0.246858	0.000101450
15	0.165092	0.165138	0.02786325	0.165091	0.000425415
20	0.123966	0.123959	0.00564671	0.123965	0.00101878

The oscillation of the particle on a rotating parabola is governed by the following equation (Nayfeh and Mook, 1995):

$$(1 + 4q^2u^2)\ddot{u} + 4q^2u\dot{u}^2 + \Lambda u = 0 \qquad (27)$$

where $\Lambda = 2gq - \Omega^2$ and the initial conditions are: $u(0) = A$ and $\dot{u}(0) = 0$. An integration of Equation (27) gives the exact period as:

$$T_{ex} = 4\Lambda^{-1/2}\int_0^A \left(\frac{1 + 4q^2u^2}{A^2 - u^2}\right)^{1/2} du \qquad (28)$$

Applying the transformation $u = A\cos\theta$ gives:

$$T_{ex} = 4\Lambda^{-1/2}\int_0^{\pi/2} \sqrt{1 + 4q^2A^2\cos^2\theta}\, d\theta \qquad (29)$$

Again, using the trigonometric identity $\cos^2\theta = 1 - \sin^2\theta$, Equation (29) can be expressed as:

$$T_{ex} = 4[(1 + 4q^2A^2)/\Lambda]^{1/2}E(m) \qquad (30)$$

where $m = 4q^2A^2/(1 + 4q^2A^2)$. Therefore, exact frequency is:

$$\omega_{ex} = \frac{\pi\sqrt{\Lambda}}{2(1 + 4q^2A^2)^{1/2}E(m)} \qquad (31)$$

From Equations (14) and (31), the approximate frequency was derived as:

$$\omega_{app} = \frac{\Lambda^{1/2}\beta\left[1 + \sqrt{\beta} + \left(8(1+\beta)\sqrt{\beta}\right)^{1/4}\right]^2}{\left(1 + \sqrt{\beta}\right)^2\left[\left(\sqrt{1+\beta} + \sqrt{8\beta^{1/2}}\right)^2 - 14\sqrt{\beta}\right]} \qquad (32)$$

where $\beta = (1 + 4q^2A^2)^{-1/2}$.

The results of Equation (32) were compared with exact results and published results (Big-Alabo and Ossia, 2020) obtained using the continuous piecewise linearization method (CPLM). The comparison is presented in Table 2, which shows that the present approximate frequency is several orders more accurate than the CPLM estimate, although the latter is also very accurate. As demonstrated in Fig. 4, the system exhibits very strong nonlinear oscillations for $A \geq 1.0$ when $q = 1.0$ and $\Lambda = 10.0$. Hence, Table 2 also demonstrates that Equation (32) is very accurate for the strong nonlinear oscillations of the system.

3.3 Geometry-related applications

In this section, it has been demonstrated that the present approximation of the $E(m)$ function is applicable to some geometry-related problems. The problems of estimating the perimeter of an ellipse and the confocal lemniscate of Bernoulli are reasonably resolved using the present $E(m)$ approximation. It is important to mention that applications of the geometric analysis of the lemniscate can be found in 2-D harmonic analysis (Boyadzhiev and Boyadzhiev, 2018), fashion design, interior decoration and arts. For example, the "perfect" bow tie is a lemniscate. On the other hand, the geometric analysis of the ellipse has application in the estimation of the sizes and orbital distances of planetary bodies (Michon, 2020), design of elliptic reflectors, phase visualization in electronic signal analysis, wave propagation in transversely orthotropic composite plates (Olsson, 1992), design of elliptical gears (Bair, 2004) and design of semi-elliptical leaf springs (Solanki and Kaviti, 2018).

3.3.1 Perimeter of an ellipse

The equation of an ellipse with semi-major length, a, and semi-minor length, b, is given as:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{33}$$

The corresponding parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. The length of an elemental arc of the ellipse is described by the differential: $ds=\sqrt{(dx)^2+(dy)^2}$. Substituting the parametric equations into the differential gives: $ds = \sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$. Integrating from $\theta = 0$ to $\theta=\pi/2$ produces the perimeter of a quarter ellipse. Therefore, the perimeter of an ellipse is given by the integral:

$$P=4\int_0^{\pi/2}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta \tag{34}$$

With simple algebraic manipulation equation (34) can be transformed into:

$$P=4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\,d\theta=4aE(m) \tag{35}$$

where $m=e^2=1-b^2/a^2$ and e is the eccentricity of the ellipse bounded to the range $0\le e\le 1$. A value of $e=0$ represents a circle of radius a, while a value of $e=1$ represents a degenerate ellipse, the limit of the so-called flat ellipse, having a length of $2a$ and a perimeter, $P = 4a$.

From Equations (14) and (35) the approximate perimeter of an ellipse can be calculated as:

$$P_{app}=\frac{2\pi a\left(1+\sqrt{\beta}\right)^2\left[1+\beta-6\sqrt{\beta}+\left(32(1+\beta)\sqrt{\beta}\right)^{1/2}\right]}{\left[1+\sqrt{\beta}+\left(8(1+\beta)\sqrt{\beta}\right)^{1/4}\right]^2} \tag{36}$$

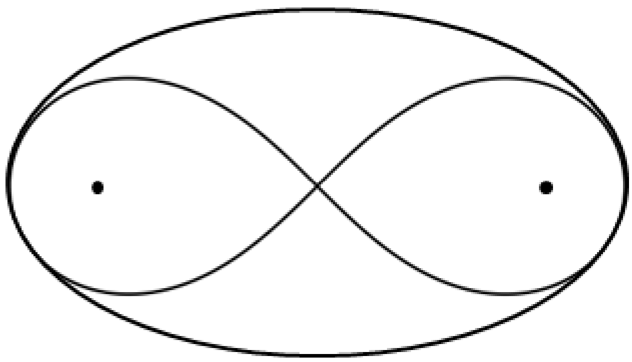

Fig. 5: A lemniscate circumscribed by a confocal self-complementary ellipse.

where $\beta=b/a=\sqrt{1-m}$ is the ratio of the semi-minor length to the semi-major length.

In 1914, the Indian mathematician, Srinivasa Ramanujan, proposed an approximate formula for the perimeter of the ellipse as (Michon, 2020):

$$P_{app}=\pi(a+b)\left(1+\frac{3h}{10+\sqrt{4-3h}}\right) \tag{37}$$

where
$h=(a-b)^2/(a+b)^2=(1-\beta)^2/(1+\beta)^2$.

A comparison between the present approximate formula for the perimeter of an ellipse with Ramanujan's formula shows that both formulae give similar results for $0\le e\le 0.665$ with a maximum RE that is less than $1.0\times10^{-11}\%$. For

$0.666 \le e \le 0.998$, the present formula is several orders more accurate than Ramanujan's formula. However, Ramanujan's formula is more accurate for $e = 1.0$ but this is a trivial case since its solution is known and does not require a special formula.

3.3.2 Perimeter of a lemniscate

The lemniscate (see Fig. 5), also called the lemniscate of Bernoulli, is a geometrical plane curve whose equation is defined by the polar function (Stroud and Booth, 2001): $r^2 = a^2 \cos 2\theta$ where $a \neq 0$. The lemniscate has a width of $2a$ and holds true for the angle ranges $-\pi/4 \le \theta \le \pi/4$ and $3\pi/4 \le \theta \le 5\pi/4$. A special case when $a = 1$ gives a lemniscate that has a focal distance of $\sqrt{2}$. The perimeter of this lemniscate is of historical significance as its determination by Carl Friedrich Gauss lead to the first connection between the AGM and elliptic integrals in 1799. Adlaj (2012) demonstrated that the special lemniscate is confocal (i.e. has the same focal points) with a self-complementary ellipse having a unit semi-major length (i.e. $e = 1/\sqrt{2}$). Hence, it would be interesting to estimate the ratio of their perimeters.

The length of a polar curve bounded by radius vectors inclined at θ_1 and θ_2 (where $\theta_2 > \theta_1$) is defined by the following integral (Stroud and Booth, 2001):

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (38)$$

Substituting $r^2 = \cos 2\theta$ in Equation (38) and integrating between $\theta_1 = 0$ and $\theta_2 = \pi/4$, the perimeter of the special lemniscate can be expressed as:

$$P = 4\int_0^{\pi/4} \frac{d\theta}{\sqrt{1 - 2\sin^2\theta}} \qquad (39)$$

After a simple transformation using $2\sin^2\theta = \sin^2\varphi$, we arrive at:

$$P = 2\sqrt{2}\int_0^{\pi/2} \left(1 - \frac{1}{2}\sin^2\varphi\right)^{-1/2} d\varphi \qquad (40)$$

The integral in Equation (40) is the $K(m)$ function with $m = 1/2$. Therefore, $P = 2\sqrt{2}K(1/2)$ which can also be written as $P = \sqrt{2}\pi/M(1,1/\sqrt{2})$. Using either Equation (10) or (25), the perimeter of the lemniscate was calculated as 5.244115108584240 to 16 significant figures with an RE of 1.1976×10^{-14}% compared to the exact value. Also, using Equation (36), the perimeter of the confocal self-complementary ellipse was calculated as 5.402575524190701 with an RE of 1.40916×10^{-14}%. Hence, the ratio of the perimeter of the confocal self-complementary ellipse to the perimeter of the circumscribed lemniscate is 1.030216807283096. The exact ratio of the perimeters of the confocal self-complementary ellipse and the circumscribed ellipse was derived as $\sqrt{2}E(1/2)/K(1/2)$ which means that the present approximation has a relative error of 0.0%.

4. Conclusion

A new algebraic formula for computing the $E(m)$ function was derived using four iterations of the AGM and MAGM algorithms and their connection to the $E(m)$ function. The formula was evaluated for a wide range of m-values and found to be accurate to less than 5.1×10^{-4}% relative error for $-1000 \le m \le 0.999$. The accuracy of the present $E(m)$ formula was shown to surpass the polynomial formula by Abramowitz and Stegun (1972) and Ramanujan's formula for calculating the perimeter of an ellipse. The present $E(m)$ formula can be used to derive very accurate periodic solutions for the nonlinear frequency of some complex nonlinear oscillators.

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Appendix

The derivation of the simplified fourth-term approximation for the MAGM algorithm is presented below.

For $n = 1$,

$$a_1 = \frac{1}{2}(a_0 + b_0)\,;\; b_1 = \sqrt{a_0 b_0}\,;\; c_1 = -\sqrt{a_0 b_0} = -b_1$$

For $n = 2$,

$$a_2 = \frac{1}{2}(a_1 + b_1) = \frac{1}{2}\left[\frac{1}{2}(a_0 + b_0) + \sqrt{a_0 b_0}\right] = \frac{1}{4}\left(\sqrt{a_0} + \sqrt{b_0}\right)^2$$

$$b_2 = \sqrt{2b_1(a_1 + b_1)} - b_1 = (a_0 b_0)^{1/4}\left(\sqrt{a_0} + \sqrt{b_0}\right) - \sqrt{a_0 b_0}$$

$$c_2 = -\sqrt{2b_1(a_1 + b_1)} - b_1 = -(a_0 b_0)^{1/4}\left(\sqrt{a_0} + \sqrt{b_0}\right) - \sqrt{a_0 b_0}$$

For $n = 3$,

$$a_3 = \frac{1}{2}(a_2 + b_2) = \frac{1}{8}\left[\left(\sqrt{a_0} - \sqrt{b_0}\right)^2 + 4(a_0 b_0)^{1/4}\left(\sqrt{a_0} + \sqrt{b_0}\right)\right]$$

$$b_3 = c_2 + \sqrt{(a_2 - c_2)(b_2 - c_2)}$$

$$= -(a_0 b_0)^{1/4}\left(\sqrt{a_0} + \sqrt{b_0}\right) - \sqrt{a_0 b_0} + \frac{\sqrt{2}}{2}\left(a_0^{1/4} + b_0^{1/4}\right)^2 (a_0 b_0)^{1/8}\left(\sqrt{a_0} + \sqrt{b_0}\right)^{1/2}$$

Because our goal is to get the expression for the fourth-term approximation, i.e. a_4, we do not need c_3 and consequently, it was not considered. Then, for $n = 4$,

$$a_4 = \frac{1}{2}(a_3 + b_3)$$

$$= \frac{1}{16}\left[\left(\sqrt{a_0} - \sqrt{b_0}\right)^2 - 4(a_0 b_0)^{1/4}\left(\sqrt{a_0} + \sqrt{b_0}\right) - 8\sqrt{a_0 b_0} + 4\sqrt{2}\left(a_0^{1/4} + b_0^{1/4}\right)^2 (a_0 b_0)^{1/8}\left(\sqrt{a_0} + \sqrt{b_0}\right)^{1/2}\right]$$

Since, $4(a_0 b_0)^{1/4}\left(\sqrt{a_0} + \sqrt{b_0}\right) + 8\sqrt{a_0 b_0} = 4(a_0 b_0)^{1/4}\left(a_0^{1/4} + b_0^{1/4}\right)^2$ then,

$$a_4 = \frac{1}{16}\left(a_0^{1/4} + b_0^{1/4}\right)^2 \left[\left(a_0^{1/4} - b_0^{1/4}\right)^2 - 4(a_0 b_0)^{1/4} + 4\sqrt{2}(a_0 b_0)^{1/8}\left(\sqrt{a_0} + \sqrt{b_0}\right)^{1/2}\right]$$

Now, $\left(a_0^{1/4} - b_0^{1/4}\right)^2 - 4(a_0 b_0)^{1/4} = \sqrt{a_0} + \sqrt{b_0} - 6(a_0 b_0)^{1/4}$. Therefore,

$$a_4 = \frac{1}{16}\left(a_0^{1/4} + b_0^{1/4}\right)^2 \left\{\left[\left(\sqrt{a_0} + \sqrt{b_0}\right)^{1/2} + \sqrt{8(a_0 b_0)^{1/4}}\right]^2 - 14(a_0 b_0)^{1/4}\right\}$$

This final expression for a_4 is the same as Equation (12).