Uniport Journal of Engineering and Scientific Research (UJESR)

Vol. 6, Special Issue, 2021, Page 44-49
ISSN: 2616-1192

(www.ujesr.org)

An Alternate Formula for the Period of a Large-Angle Pendulum

Big-Alabo, A*[1]
[1]Applied Mechanics & Design (AMD) Research Group
Department of Mechanical Engineering, Faculty of Engineering, University of Port Harcourt, Port Harcourt, Nigeria
*Corresponding author's email: akuro.big-alabo@uniport.edu.ng

Abstract
This paper presents an explicit formula for estimate of the pendulum period. The formula was derived based on the arithmetic-geometric mean (AGM) approximation for the complete elliptic integral of the first kind. The results obtained using the present formula were found to be very accurate for $0° < A < 180°$ *with a maximum relative error of 0.0342% at* $A = 179.9°$. *The formula can be implemented with a pocket calculator and is therefore recommended for use in undergraduate physics and mechanics courses.*
Keywords: *Pendulum, arithmetic-geometric mean, elliptic integral, geometric nonlinearity, large-amplitude oscillations*

Received: 30th June, 2022 *Accepted: 17th July, 2022*

1. Introduction

The pendulum is notable in introductory physics and mechanics courses because of its pedagogical value. For one, the pendulum can be used to introduce nonlinear oscillation concepts (Big-Alabo, 2020a). Also, the pendulum can be used to estimate local gravity in simple classroom laboratory experiments (Oliveira, 2016) and to study the periodic motion of many physical systems governed by pendulum-like motion (Lima, 2008).

The model for the undamped motion of the pendulum is well-known and can be written as:

$$\ddot{\varphi} + \omega_0^2 \sin\varphi = 0 \qquad (1)$$

For the mathematical pendulum, $\omega_0 = \sqrt{g/l}$ where g is the acceleration due to gravity and l is the length of the pendulum. For other pendulum-like motions, ω_0 is expressed differently and is based on the system's parameters. The initial conditions to Equation (1) are $\varphi(0) = A$ and $\dot{\varphi}(0) = 0$ where $A \in [0, \pi]$ is the oscillation amplitude in radians.

Equation (1) has a trigonometric nonlinearity arising from geometric effects. During small-angle oscillations, i.e. $A < 10°$, the geometric nonlinear effect is negligible and we can safely assume that $\sin\varphi = \varphi$. Therefore, the pendulum motion is essentially governed by a linear differential equation (i.e. $\ddot{\varphi} + \omega_0^2\varphi = 0$) with a constant period, $T_0 = 2\pi/\omega_0$. On the other hand, during large-angle oscillations, the period depends on the amplitude due to geometric nonlinear effect and the exact period is given in terms of the complete elliptic integral of the first kind as (Lima, 2008):

$$\frac{T_{ex}}{T_0} = \frac{2}{\pi} K(k^2) \qquad (2)$$

where $K(k^2) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\theta}} d\theta$ is the complete elliptic integral of the first kind and $k = \sin(A/2)$. Since $K(k^2)$ cannot be evaluated exactly in terms of elementary functions, it is normally evaluated numerically. Alternatively, approximate solutions for the pendulum period are available in terms of elementary functions (Big-Alabo, 2020a; Lima, 2008; Benacka, 2017; Kidd and Fogg, 2002; Johannessen, 2010; Belendez et al, 2009; Hite, 2005; Big-Alabo, 2020b; Carvalhaes and Suppes, 2008). However, most of these approximations are only accurate to less than 1.0% relative error for a limited range of the possible amplitudes and would require complementary solutions for those amplitudes in which they are inaccurate (Big-Alabo, 2020b).

Accurate time period formulae that are explicit functions of amplitude and can be easily computed using a pocket calculator are important for introductory physics and mechanics courses. In this context, the time period formula is considered accurate if it gives less than 1.0% relative error for $0° < A \leq 179°$. Relatively few studies (Big-Alabo, 2020b; Carvalhaes and Suppes, 2008) have attempted to provide an explicit formula that produces less than 1.0% relative error for $0° < A \leq$

179°. The formula by Big-Alabo (2020b) was based on quintication of the restoring force while the formula of Carvalhaes and Suppes (2008) was based on fourth-order AGM method. The fourth-order AGM formula remains one of the most accurate to date but requires spreadsheet implementation because it is too lengthy for a pocket calculator computation (Carvalhaes and Suppes, 2008). In this article, a new fourth-order AGM formula that has the same accuracy as the formula of Carvalhaes and Suppes (2008) but is simple enough to allow pocket calculator computation is proposed based on trigonometric simplification of the AGM method. Because the present method relies on the use of trigonometric identities, it can be easily derived by undergraduates. The present method also provides an interesting way of applying the AGM to the pendulum problem.

2. AGM method for estimating pendulum period

Carvalhaes and Suppes (2008) derived expressions for the first four terms of the AGM sequence in terms of elementary functions (see Equations (A.1) to (A.4) in appendix) and demonstrated that the fourth term approximation for the pendulum period (Equation (A.4)) gives a relative error of less than 1.0% for $0 < A \leq 179.99°$. However, Equation (A.4) is difficult to implement using a pocket calculator and is better suited for spreadsheet implementation. In this section, trigonometric identities were applied to simplify the AGM approximation in order to derive a formula that can be easily implemented using a pocket calculator.

The AGM is a recursive algorithm of two sequences; an arithmetic sequence and a geometric sequence. The sequences are defined as shown:

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) \qquad (3a)$$

$$b_n = \sqrt{a_{n-1}b_{n-1}} \qquad (3b)$$

Therefore, the AGM of two numbers a and b such that $a > b > 0$ can be defined as:

$$M(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \qquad (4)$$

where $a_0 = a$ and $b_0 = b$ are the initial values used in the iteration.

An important feature of this simple algorithm that makes it attractive is its quadratic convergence (Carvalhaes and Suppes, 2008). Its relationship to the elliptic integral of the first kind has been long established and can be expressed as (Carvalhaes and Suppes, 2008):

$$K(k^2) = \frac{\pi}{2M\left(1, \sqrt{1-k^2}\right)} \qquad (5)$$

Since $k = \sin(A/2)$ for the simple pendulum, then equation (2) can be rewritten as:

$$T_{ex} = \frac{T_0}{M(1, \cos(A/2))} \qquad (6)$$

where $a_0 = 1$ and $b_0 = \cos(A/2)$.

Now, we can turn our attention to deriving the fourth term approximation of $M(1, \cos(A/2))$ in simplified form. In this case, $a_0 = 1$ and $b_0 = \cos(A/2)$. Therefore, the AGM sequence up to the fourth term (a_4) was derived using Equations (3a,b) as shown in Table 1. In simplifying the expression

Table 1: First four terms of the AGM sequence

n	a_n	b_n
1	$\frac{1}{2}\left[1 + \cos\left(\frac{A}{2}\right)\right] = \cos^2\left(\frac{A}{4}\right)$	$\sqrt{\cos\left(\frac{A}{2}\right)} = \cos^{\frac{1}{2}}\left(\frac{A}{2}\right)$
2	$\frac{1}{4}\left[1 + \cos\left(\frac{A}{2}\right) + 2\cos^{\frac{1}{2}}\left(\frac{A}{2}\right)\right] = \frac{1}{4}\left[1 + \cos^{\frac{1}{2}}\left(\frac{A}{2}\right)\right]^2$	$\sqrt{\cos^2\left(\frac{A}{4}\right)\cos^{\frac{1}{2}}\left(\frac{A}{2}\right)} = \cos\left(\frac{A}{4}\right)\cos^{\frac{1}{4}}\left(\frac{A}{2}\right)$
3	$\frac{1}{4}\left\{\cos^2\left(\frac{A}{4}\right) + \cos^{\frac{1}{2}}\left(\frac{A}{2}\right) + 2\cos\left(\frac{A}{4}\right)\cos^{\frac{1}{4}}\left(\frac{A}{2}\right)\right\} = \frac{1}{4}\left[\cos\left(\frac{A}{4}\right) + \cos^{\frac{1}{4}}\left(\frac{A}{2}\right)\right]^2$	$\sqrt{\frac{1}{4}\left[1 + \cos^{\frac{1}{2}}\left(\frac{A}{2}\right)\right]^2 \cos\left(\frac{A}{4}\right)\cos^{\frac{1}{4}}\left(\frac{A}{2}\right)} = \frac{1}{2}\left[1 + \cos^{\frac{1}{2}}\left(\frac{A}{2}\right)\right]\cos^{\frac{1}{2}}\left(\frac{A}{4}\right)\cos^{\frac{1}{8}}\left(\frac{A}{2}\right)$

4	$\frac{1}{16}\Big\{2\cos^2\left(\frac{A}{4}\right)+2\cos^{\frac{1}{2}}\left(\frac{A}{2}\right)+4\cos\left(\frac{A}{4}\right)\cos^{\frac{1}{4}}\left(\frac{A}{2}\right)+4\left[1+\cos^{\frac{1}{2}}\left(\frac{A}{2}\right)\right]\cos^{\frac{1}{2}}\left(\frac{A}{4}\right)\cos^{\frac{1}{8}}\left(\frac{A}{2}\right)\Big\}$ $=\frac{1}{16}\left[1+\cos^{\frac{1}{2}}\left(\frac{A}{2}\right)+2\cos^{\frac{1}{2}}\left(\frac{A}{4}\right)\cos^{\frac{1}{8}}\left(\frac{A}{2}\right)\right]^2$	Not necessary for the fourth term approximation

for a_n the trigonometric identity $\cos^2\theta = (1+\cos 2\theta)/2$ was applied. Also, $(\cos\theta)^x = \cos^x\theta$ was used.

From the last row of Table 1 and using Equation (3), we can write:

$$M\left(1,\cos\frac{A}{2}\right) \approx \frac{1}{16}\left[1+\cos^{\frac{1}{2}}\left(\frac{A}{2}\right)+2\cos^{\frac{1}{2}}\left(\frac{A}{4}\right)\cos^{\frac{1}{8}}\left(\frac{A}{2}\right)\right]^2 \quad (7)$$

Substituting Equation (7) in (6) gives the approximate period of the pendulum, T_{app}, as:

$$\frac{T_{app}}{T_0} = \frac{16}{\left[1+\cos^{\frac{1}{2}}\left(\frac{A}{2}\right)+2\cos^{\frac{1}{2}}\left(\frac{A}{4}\right)\cos^{\frac{1}{8}}\left(\frac{A}{2}\right)\right]^2} \quad (8)$$

Equation (8) is compact and simple compared to Equation (A.4) derived by Carvalhaes and Suppes (2008). The derivation of Equation (8) can be easily demonstrated during classroom session because it only involves the application of trigonometric identities and factorisation, which are skills already acquired from high school and first year mathematics. Hence, deriving the present formula would be an interesting exercise where students can apply basic mathematical skills to derive the solution to a real-world problem capable of giving them an early introduction to the nonlinear world.

A close look at the expressions in Table 1 reveals the following relationship:

$$a_n = \frac{1}{4}\left(\sqrt{a_{n-2}}+\sqrt{b_{n-2}}\right)^2 \qquad \text{for } n \geq 2 \qquad (9)$$

Therefore, the second and higher approximations can be obtained by simply using Equation (9). Consequently, another interesting exercise for students would be to use Equation (9) to derive the 5th order approximation and compare the results obtained with those of Equation (8) for $0° < A \leq 179.99°$.

3. Results and discussion

Table 2 shows the computed results for $1.0° \leq A \leq 179.99°$ obtained using Equation (8) and compared with the exact period and the formula by Big-Alabo (2020a). The present formula produces a maximum relative error of 0.00000% for $A \leq 175°$, 0.00103% for $A \leq 179°$, 0.03428% for $A \leq 179.9°$ and 0.25485% for $A \leq 179.99°$. These results are several orders more accurate than those of the formula by Big-Alabo (2020b) except for $A = 179.99°$ where the present formula is about twice more accurate.

Fig. 1 is a plot of the normalized period versus amplitude while Fig. 2 shows the

Table 2: Numerical results of exact and approximate time period for the pendulum

		Big-Alabo (2020b)		This study	
A (deg)	T_{ex}/T_0	T_{app}/T_0	% Error	T_{app}/T_0	% Error
1	1.00002	1.00002	0.00000	1.00002	0.00000
5	1.00048	1.00048	0.00000	1.00048	0.00000
10	1.00191	1.00191	0.00000	1.00191	0.00000
20	1.00767	1.00767	0.00000	1.00767	0.00000
30	1.01741	1.01741	0.00000	1.01741	0.00000
40	1.03134	1.03134	0.00000	1.03134	0.00000
50	1.04978	1.04978	0.00000	1.04978	0.00000
60	1.07318	1.07318	0.00000	1.07318	0.00000
70	1.10214	1.10214	0.00000	1.10214	0.00000
80	1.13749	1.13748	0.00088	1.13749	0.00000
90	1.18034	1.18031	0.00254	1.18034	0.00000

100	1.23223	1.23216	0.00568	1.23223	0.00000
110	1.29534	1.29520	0.01081	1.29534	0.00000
120	1.37288	1.37262	0.01894	1.37288	0.00000
130	1.46982	1.46933	0.03334	1.46982	0.00000
140	1.59445	1.59354	0.05707	1.59445	0.00000
150	1.76220	1.76052	0.09534	1.7622	0.00000
160	2.00751	2.00428	0.16090	2.00751	0.00000
170	2.43935	2.43270	0.27302	2.43936	0.00000
175	2.87766	2.86737	0.35758	2.87766	0.00000
179	3.90107	3.88464	0.42117	3.90103	0.00103
179.9	5.36687	5.34567	0.39502	5.36503	0.03428
179.99	6.83274	6.79576	0.54120	6.81532	0.25495

corresponding error analysis. The error was calculated as $100 \times \left|1 - T_{app}/T_{ex}\right|\%$. These figures show an excellent agreement between the present formula, exact formula and the formula by Big-Alabo (2020b). However, the error plot clearly shows the superior accuracy of the present formula in comparison with the formula by Big-Alabo (2020b).

4. Conclusion

Many oscillating systems in physics and engineering undergo pendulum-like motion. In some cases (e.g. elliptic filters and in quantum mechanics), it is necessary to estimate the period of the large-angle motion accurately. In this paper, an accurate formula that is based on elementary functions has been derived to estimate the period of a pendulum. The present formula is based on trigonometric simplification of the fourth-order approximation to the AGM formula for the exact period of the pendulum. The present formula can be easily computed with a pocket calculator and it produces a maximum relative error of 0.0342% for $A \leq 179.9°$. Also, checks confirmed that the present formula produces exactly the same results as Equation (A.4). This is not surprising since both equations are based on the fourth term approximation of $M\left(1, \cos\frac{A}{2}\right)$.

In addition, the AGM sequence relation defined by Equation (9) can be applied to easily obtain very accurate elementary function approximations to the

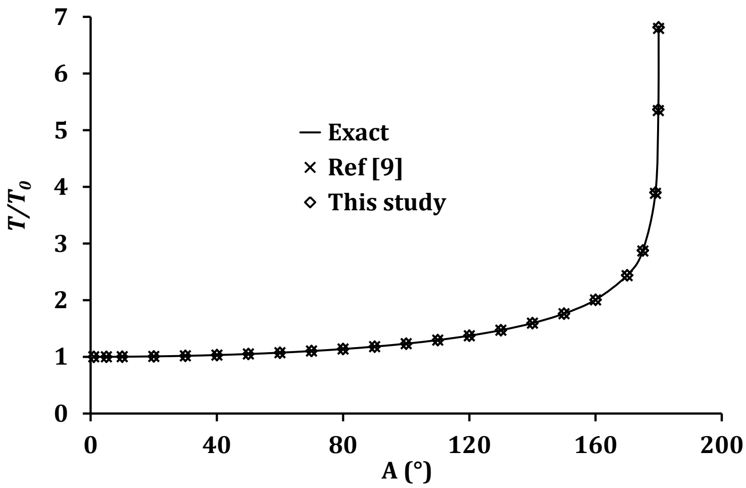

Fig. 1: Time period of pendulum for the range: $1° \leq A \leq 179.9°$. Note that Ref [9] is the pendulum formula of Big-Alabo (2020b).

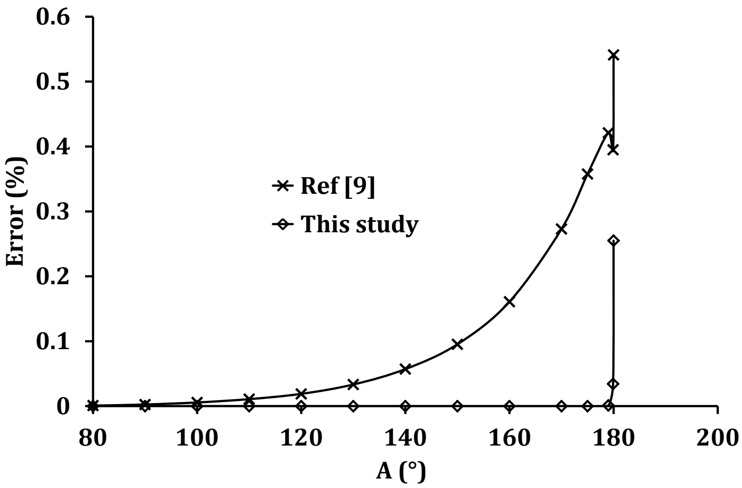

Fig. 2: Error analysis for the present formula and published formula (Big-Alabo, 2020b): $80° \leq A \leq 179.99°$.

elliptic integral of the first kind, when the arguments are defined differently from that of the pendulum. This has the advantage that elliptic integrals can be introduced in undergraduate courses in a manner that only requires simple algebraic manipulations.

Funding
None received.

Acknowledgement
This article was written in honour of Professor Mike O. Onyekonwu on the occasion of his 70th birthday and retirement from the University of Port Harcourt after a meritorious service of over four decades.

References

Belendez, A., Rodes, J. J., Belendez, T., Hernandez, A. (2009): Approximation for a large-angle simple pendulum period, European Journal of Physics, 30, L25–L28.

Benacka, J. (2017): Fast converging exact power series for the time and period of the simple pendulum, European Journal of Physics, 38, 025004.

Big-Alabo, A. (2020a): Approximate periodic solution for the large-amplitude oscillations of a simple pendulum, Int'l Journal of Mechanical Engineering Education, 48(4):335-350.

Big-Alabo, A. (2020b): Approximate period for the large-amplitude oscillations of the simple pendulum based on quintication of the restoring force, European Journal of Physics, 41(1), 015001, 10pp.

Carvalhaes, C. G. and Suppes, P. (2008): Approximations for the period of the simple pendulum based on the arithmetic-geometric mean, American Journal of Physics, 76:1150–1154.

Hite, G. E. (2005): Approximations for the period of the simple pendulum, Physics Teacher, 43: 290-292.

Johannessen, K. (2010): An approximate solution to the equation of motion for large-angle oscillations of the simple pendulum with initial velocity, European Journal of Physics, 31:511-518.

Kidd, R. B. and Fogg, S. L. (2002): A simple formula for the large-angle pendulum period, Physics Teacher, 40:81-83.

Lima, F. M. S. (2008): Simple 'log formulae' for pendulum motion valid for any amplitude, European Journal of Physics, 29:1091-1098.

Oliveira, V. (2016): Measuring g with a classroom pendulum using changes in the pendulum string length, Physics Education, 51, 063007, 3 pages.

Appendix

Carvalhaes and Suppes (2008) derived time period estimates of the pendulum based on the first four terms of the AGM as shown:

$$\frac{T_1}{T_0} = \frac{2}{1 + \cos\left(\frac{A}{2}\right)} \qquad (A.1)$$

$$\frac{T_2}{T_0} = \frac{4}{1 + \cos\left(\frac{A}{2}\right) + 2\cos^{1/2}\left(\frac{A}{2}\right)} \qquad (A.2)$$

$$\frac{T_3}{T_0} = \frac{8}{1 + \cos\left(\frac{A}{2}\right) + 2\cos^{1/2}\left(\frac{A}{2}\right) + 2^{3/2}\cos^{1/4}\left(\frac{A}{2}\right)\left[1 + \cos\left(\frac{A}{2}\right)\right]^{1/4}} \qquad (A.3)$$

$$\frac{T_4}{T_0} = \frac{16}{\left\{\begin{matrix} 1 + \cos\left(\frac{A}{2}\right) + 2\cos^{\frac{1}{2}}\left(\frac{A}{2}\right) + 2^{\frac{3}{2}}\cos^{\frac{1}{4}}\left(\frac{A}{2}\right)\left[1 + \cos\left(\frac{A}{2}\right)\right]^{\frac{1}{4}} + \\ 2^{\frac{7}{4}}\cos^{\frac{1}{8}}\left(\frac{A}{2}\right)\left[1 + \cos\left(\frac{A}{2}\right)\right]^{\frac{1}{4}}\left[1 + \cos\left(\frac{A}{2}\right) + 2\cos^{\frac{1}{2}}\left(\frac{A}{2}\right)\right]^{\frac{1}{2}} \end{matrix}\right\}} \qquad (A.4)$$