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Development of a Three-Step Non-Hybrid Block Extended Second Derivative Backward Differentiation Formula

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Abstract
This paper present non-hybrid block scheme of order six for solving stiff ordinary differential equation. This was done by construction of continuous formulation through interpolation and collocation of first and second derivative function with power series as a basic function. The activeness of the new constructed numerical scheme was tested on some numerical problems to show the accuracy, efficiency and error bound. The approximate results are compared with existing method where the desire accuracy of the constructed method performed better than ones compared.
Keywords: *Block Method, Non-Hybrid, Second Derivative, Backward Differentiation Formula, Stiff ODEs*
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1. Introduction

In this research paper, the construction of three-step numerical method for approximate solution of first order stiff ordinary differential equations would be considered. Currently, higher order ordinary differential equations are solved by a reduction to a system of first order ordinary differential equations Fatunla (1988) Jator (2007). In order to overcome the difficulties, block method was developed Fatunla (1991). Continuous collocation and interpolation technique are now widely used for the derivation of linear multistep methods (LMMs) and hybrid block methods. Many continuous LMMs have been derived using different techniques and approaches for solving first order stiff systems Alabi (2008). The Dahlquist barrier theorem was overcome by many authors who developed modified forms of linear multistep methods called hybrid by working for off-grid points in construction process in determining good scheme Gear (1965). The derivation of two-step continuous and discrete LMMs using power series as basic functions yield a better numerical result Okunuga and Ehigie (2009).

Developed a high order block implicit multistep methods have been established to be A-stable (Skwame et al., 2019). Constructed a linear multistep method with continuous coefficients and used it to obtain multiple finite difference methods which were directly applied to solve first-order ODEs Mohammed (2011). For authors to get out of the difficulties in solving heavy mathematical problems and block method was proposed. This scheme generates better approximation at more than one grid point simultaneously Kuboye and Omar (2015). Constructed a continuous linear multistep method with interpolation and collocation for the approximate solution of first-order ODE with constants step size (Odekunle et al., 2012). The linear multistep methods (LMMs) gives advantage of generating discrete method would be be using to solve stiff first-order ordinary differential equations (Bakari et al., 2018). Hybrid block methods of higher order and the system of first order ODEs have been constructed by many researchers and we were motivated by their work to develop the non-hybrid block method for system of first order ODEs.

2. Methodology

The concept is to approximate the analytic solution $y(x)$ in the partition $\pi[a,b]$=$[a = x_0 < x_1 < \cdots < x_n = b]$ of the integration interval$[a,b]$ by a power series polynomial of the form function below.

$$y(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_p x^p = \sum_{j=0}^{p} a x^j \quad (1)$$

Where $p = r + s + t - 1$

Equation (1) is twice-continuously differentiable function of $y(x)$.

$$y'^{(x)} = \sum_{j=0}^{r+s+t-1} j a_j x^{j-1} \quad (2)$$

Now, differentiate the function $y(x)$ again

$$y''^{(x)} = \sum_{j=0}^{r+s+t-1} j(1-j)a_j x^{j-2} \quad (3)$$

where$x \in [a,b]$, the $a's$ are real unknown parameters to be determined and $r+s+t$ is the sum of the number of collocation and interpolation points. Interpolation points:

$$y(x) = \sum_{j=0}^{6} a_j x^j = y_{n+j} \quad (4)$$

Collocation points @ first derivative

$$y'(x) = \sum_{j=0}^{6} j a_j x^{j-1} = f_{n+j} \quad (5)$$

Collocation points @ second derivative

$$y''(x) = \sum_{j=0}^{6} j(j-1)a_j x^{j-2} = g_{n+j} \quad (6)$$

In this case $k=3$ of this method gives interpolation points at $x = x_n, x_{n+1}, x_{n+2}$ and collocating points at $x = x_{n+2}, x_{n+3}$ to give the system of equation written in matrix form $AX = B$ as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 \\ 0 & 1 & 4 & 12 & 32 & 80 & 192 \\ 0 & 1 & 6 & 27 & 108 & 405 & 1{,}458 \\ 0 & 0 & 2 & 12 & 48 & 120 & 480 \\ 0 & 0 & 2 & 18 & 108 & 405 & 2{,}430 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ f_{n+2} \\ f_{n+3} \\ g_{n+2} \\ f_{n+3} \end{bmatrix} \quad (7)$$

Solving for,a_j j =0(1)6 in Equation (7) using Gaussian elimination method and substituting into Equation (1) gives a linear multistep method with continuous coefficients function below in the form.

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + h[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+1} + \gamma_2(x)g_{n+2} + \gamma_3(x)g_{n+3}] \quad (8)$$

Using Maple software and inverting the matrix in (7) gives the elements of$A^{-1`}$.

The columns of $A^{-1`}$ give the continuous coefficients of Equation (8) as:

$$\alpha_0(x) = 1 - \frac{81x}{26h} + \frac{5049x^2}{1300h^2} - \frac{1299x^3}{520h^3} + \frac{2283x^4}{2600h^4} - \frac{417x^5}{2600h^5} + \frac{31x^6}{2600h^6}$$

$$\alpha_1(x) = \frac{216x}{13h} - \frac{11412x^2}{325h^2} + \frac{1906x^3}{65h^3} - \frac{3927x^4}{325h^4} + \frac{798x^5}{325h^5} - \frac{64x^6}{325h^6}$$

$$\alpha_2(x) = -\frac{27x}{2h} + \frac{3123x^2}{100h^2} - \frac{1073x^3}{40h^3} + \frac{2241x^4}{200h^4} - \frac{459x^5}{200h^5} + \frac{37x^6}{200h^6}$$

$$\beta_2(x) = \frac{108x}{13} - \frac{12231x^2}{650h} + \frac{3981x^3}{260h^2} - \frac{7477x^4}{1300h^3} + \frac{1322x^5}{1300h^4} - \frac{89x^6}{1300h^5}$$

$$\beta_3(x) = -\frac{40x}{13} + \frac{2772x^2}{325h} - \frac{586x^3}{65h^2} + \frac{1487x^4}{325h^3} - \frac{363x^5}{325h^4} + \frac{34x^6}{325h^5}$$

$$\gamma_2(x) = \frac{81x}{13} - \frac{5274x^2}{325} + \frac{4133x^3}{260h} - \frac{9641x^4}{1300h^2} + \frac{2159x^5}{1300h^3} - \frac{817x^6}{1300h^4}$$

$$\gamma_3(x) = -\frac{12x}{13} + \frac{842x^2}{325h} - \frac{181x^3}{65h^2} + \frac{939x^4}{650h^3} - \frac{118x^5}{325h^4} + \frac{23x^6}{650h^5}$$

Evaluating Equation (8) at the following points x_n, x_{n+1} *and* x_{n+3} yields the following discrete methods which constitute the new three- step non-hybrid block method.

$$y_n - \frac{5072}{561}y_{n+1} + \frac{4511}{561}y_{n+2} = \frac{3h}{561}[1359f_{n+2} + 616f_{n+3}] + \frac{2h^2}{5049}[325g_n - 10548g_{n+2} - 1684g_{n+3}]$$

$$y_n + \frac{848}{101}y_{n+1} - \frac{949}{101}y_{n+2} = -\frac{6h}{101}[127f_{n+2} + 48f_{n+3}] + \frac{2h^2}{303}[325g_n - 919g_{n+2} - 127g_{n+3}]$$

$$y_{n+3} - \frac{1}{325}y_{n+1} - \frac{27}{25}y_{n+2} = \frac{6h}{325}[27f_{n+2} + 23f_{n+3}] + \frac{18h^2}{325}[3g_n - 10548g_{n+2} - g_{n+3}]$$

3. Analysis of the new developed method

In this section, the estimation of the order and error constant of the block with the difference equation of the form.

$[y(x); h] = \sum_{j=0}^{k} \alpha_j y_{n+j} - h\beta_k f_{n+k} - h^2\gamma_k g_{n+k}$. (9)

Assuming that $y(x)$ is sufficiently differentiable and expand the terms in (11) as a Taylor series and comparing the coefficients of h gives Equation (10).

$L[y(x); h] = c_0 y(x) + c_1 h y'^{(x)} + c_2 h^2 y''^{(x)} + \cdots + c_p h^p y^p(x) + \cdots$ (10)

Where the constants $C_p, p = 0, 1,2 \dots., j = 1,2, \dots k$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j$$

$$C_q = \frac{1}{q!}\sum_{j=0}^{k} j^q \alpha_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\lambda_j \qquad (11)$$

where $C_0 = C_1 = C_2 \dots C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$. Therefore, C_{p+2} is the error constant and $C_{p+2}h^{p+2}y^{(p+2)}(x_n)$ is the principal local truncation error at a pointx_n.

Using the concept above, the non-hybrid block methods is constructed using MAPLE 18 SOFTWARE gives the following uniform order and error constants.

Comparing the coefficient of h, according to (Skwame et al., 2019). The order p of the method is $p = [6 \quad 6 \quad 6]^T$ and the error constant are given respectively by $\left[\frac{2339}{176715}, -\frac{37}{30303}, \frac{3}{22750}\right]$.

4. Region of absolute stability of the method

The hybrid block method is said to be absolutely stable, if for a given h, all roots of the characteristic polynomial $\pi(z, h) = \rho(z) - \bar{h}\sigma(z)$, satisfied

$|z_t| < 1$. Applying the boundary locus method, after some manipulation, then substituting the stability polynomial and obtain the region of absolute stability (Omar and Abdelrahim, 2016).

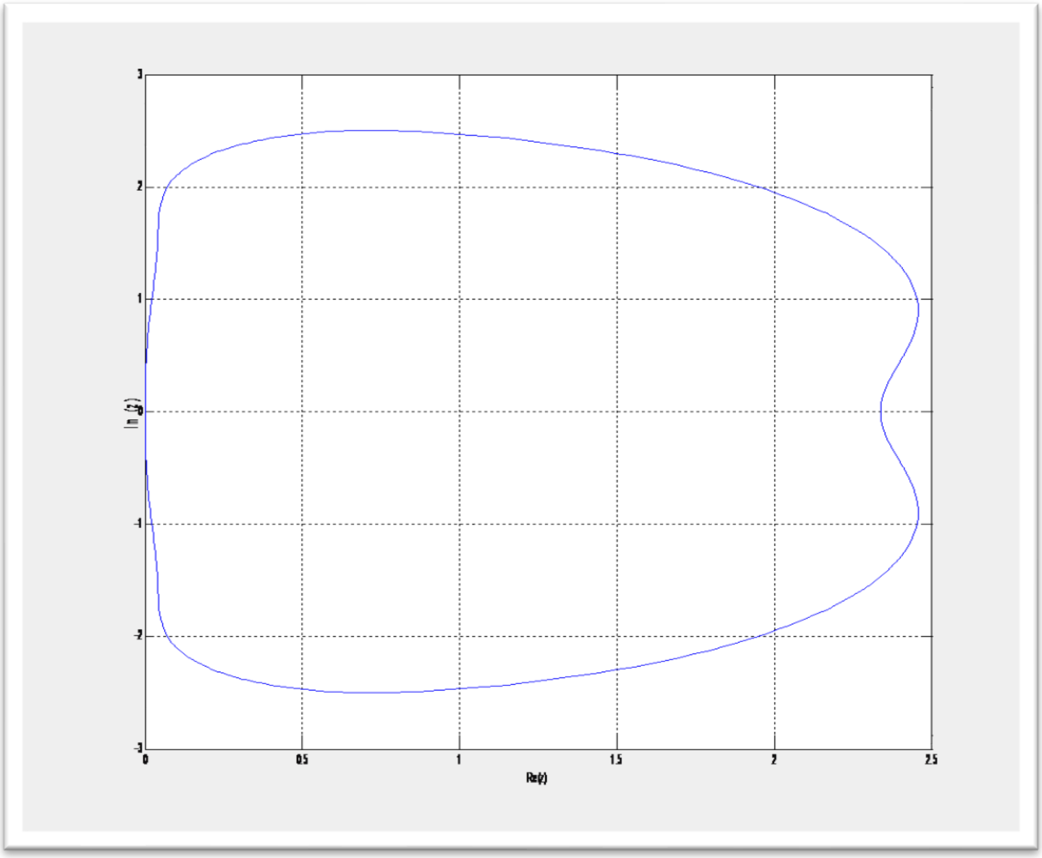

Fig. 1: Region of absolute stability of the new constructed scheme

5. Numerical examples and results

In this section, the constructed continuous non-hybrid second derivative block backward differentiation formula in block forms for step number k = 3 to solve linear and nonlinear stiff systems of ordinary differential equations.

Problem 1

$$y_1' = -29998y_1 - 59994y_2$$

$$y_2' = 9999y_1 + 19997y_2$$

Exact $y_1(x) = \left(\frac{1}{9999}\right)(29997e^{-10000x} - 19998e^{-x})$ $y_1(0) = 1$

$y_2(x) =$ -$e^{-10000x} + e^{-x}$ $y_2(0) = 0$ $with\ h = 0.1$

Problem 2

$$y_1' = -2y_1 + y_2 + 2sinx$$

$$y_2' = 998y_1 - 999y_2 + 999(cosx - sinx)$$

$with\ h = 0.1$

Exact $y_{1(x)} = 2e^{-x} + sinx$ $y_1(0) = 2$

$y_2(x) = 2e^{-x} + cosx$ $y_2(0) = 3$

Problem 3

$$y_1' = -12y_1 + 10y_2^2$$

$$y_2' = y_1 - y_2 - y_1y_2 \qquad with\ h = 0.1$$

Exact $y_{1(x)} = e^{-2x} y_1(0) = 1$

$y_2(x) = e^{-x} y_2(0) = 1$

The absolute error of numerical solutions of problem 1, problem 2 and problem 3 in Tables 1, 2, 3, 4 and 5 solved with new constructed method three-step non-hybrid block extended second derivative backward differentiation formula (3SNHBESDBDF) showed that the new method produced better results than the existing method when solving the same problem.

Table 1: Absolute errors of numerical solutions of problem 1 solved with 3SNHBESDBDF

x	**3SNHBESDBDF**(y_1)	**3SNHBESDBDF**(y_2)
0.1	8.25340E − 11	4.12450E − 11
0.2	2.35790E − 11	1.17810E − 11
0.3	5.26800E − 12	2.81700E − 12
0.4	2.50800E − 12	1.25500E − 12
0.5	1.80300E − 12	9.02000E − 13
0.6	1.06300E − 12	5.32000E − 13
0.7	4.48000E − 13	2.24000E − 13
0.8	9.90000E − 14	5.00000E − 14
0.9	2.40000E − 14	1.20000E − 14
1	8.00000E − 15	4.00000E − 15

Table 2: Absolute errors of numerical solutions of problem 2 solved with 3SNHBESDBDF

x	**3SNHBESDBDF**(y_1)	**3SNHBESDBDF**(y_2)
0.1	$-2.73555E-03$	$-1.44280E-03$
0.2	$8.91114E-03$	$9.38384E-03$
0.3	$1.67897E-02$	$1.60078E-02$
0.4	$1.08597E-02$	$9.54204E-03$
0.5	$-4.45585E-03$	$-5.09779E-03$
0.6	$-1.54544E-02$	$-1.48304E-02$
0.7	$-1.21632E-02$	$-1.08470E-02$
0.8	$2.34060E-03$	$3.13893E-03$
0.9	$1.47035E-02$	$1.42499E-02$
1	$1.35521E-02$	$1.22636E-02$

Table 3: Absolute errors of numerical solutions of problem 3 solved with 3SNHBESDBDF

x	**3SNHBESDBDF**(y_1)	**3SNHBESDBDF**(y_2)
0.1	$3.07817E-02$	$2.75475E-02$
0.2	$6.12095E-04$	$2.17811E-02$
0.3	$1.30108E-03$	$1.08293E-02$
0.4	$5.76319E-04$	$4.68079E-03$
0.5	$2.22918E-04$	$1.92320E-03$
0.6	$8.39664E-05$	$7.73942E-04$
0.7	$3.1662E-05$	$3.08326E-04$
0.8	$1.17855E-05$	$1.22092E-04$
0.9	$4.41526E-06$	$4.81376E-05$
1	$1.65452E-06$	$1.89131E-05$

Table 4: Comparison of 3SNHBESDBDF for problem 1

x	**Error in Our Method** (y_1)	$P=1$ (y_2)	**Error in Olabode & Momoh (2016)** $P=3$
0.1	$8.25340E-11$	$4.12450E-11$	4.861096E-14
0.2	$2.35790E-11$	$1.17810E-11$	9.215042E-14
0.3	$5.26800E-12$	$2.81700E-12$	1.300098E-13
0.4	$2.50800E-12$	$1.25500E-12$	1.620497E-13
0.5	$1.80300E-12$	$9.02000E-13$	1.884327E-13
0.6	$1.06300E-12$	$5.32000E-13$	2.095065E-13
0.7	$4.48000E-13$	$2.24000E-13$	2.257227E-13
0.8	$9.90000E-14$	$5.00000E-14$	2.375819E-13
0.9	$2.40000E-14$	$1.20000E-14$	2.455966E-13
1	$8.00000E-15$	$4.00000E-15$	2.502672E-13

Table 5: Comparison of 3SNHBESDBDF for problem 1

x	**Error in Our Method** (y_1)	$P = 1$ (y_2)	**Error in Jator & Li (2009)** $P = 5$
0.1	$8.25340E-11$	$4.12450E-11$	5.10704E-06
0.2	$2.35790E-11$	$1.17810E-11$	1.49586E-05
0.3	$5.26800E-12$	$2.81700E-12$	2.78532E-05
0.4	$2.50800E-12$	$1.25500E-12$	4.28908E-05
0.5	$1.80300E-12$	$9.02000E-13$	6.70307E-05
0.6	$1.06300E-12$	$5.32000E-13$	1.02637E-04
0.7	$4.48000E-13$	$2.24000E-13$	1.44907E-04
0.8	$9.90000E-14$	$5.00000E-14$	1.90905E-04
0.9	$2.40000E-14$	$1.20000E-14$	2.39733E-04
1	$8.00000E-15$	$4.00000E-15$	2.94670E-04

6. Conclusion

In this research, the constructed a three-step non-hybrid extended block second derivative backward differentiation formula for the solution of stiff ordinary differential equations, the developed scheme is of order six which was applied to solve both linear and non-linear of stiff problems. Maple and Mat lab were used to generate the scheme and numerical solutions. Three numerical experiments had been used to test the accuracy and efficiency of the developed method. The region of absolute stability of new developed method is presented in figure 1 which shows that to be zero stable, consistent and convergent.

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